Ratio and Proportion (Including Properties & Uses)

Question 1.

If a: b = 5: 3, find: $\frac{5a-3b}{5a+3b}$.

Solution:

$a:b = 5:3$

$\Rightarrow \frac{a}{b} = \frac{5}{3}$

$$\frac{5a-3b}{5a+3b} = \frac{5\left(\frac{a}{b}\right)-3}{5\left(\frac{a}{b}\right)+3} \text{ (Dividing each term by b)}$$

$$= \frac{5\left(\frac{5}{3}\right)-3}{5\left(\frac{5}{3}\right)+3}$$

$$= \frac{\frac{25}{3}-3}{\frac{25}{3}+3}$$

$$= \frac{25-9}{25+9}$$

$$= \frac{16}{34} = \frac{8}{17}$$

Question 2.

If x: y = 4: 7, find the value of (3x + 2y): (5x + y).

Solution:

$x : y = 4 : 7$

$\Rightarrow \frac{x}{y} = \frac{4}{7}$

$\frac{3x + 2y}{5x + y} = \frac{3\left(\frac{x}{y}\right) + 2}{5\left(\frac{x}{y}\right) + 1}$ (Dividing each term by y)

$= \frac{3\left(\frac{4}{7}\right) + 2}{5\left(\frac{4}{7}\right) + 1}$

$= \frac{\frac{12}{7} + 2}{\frac{20}{7} + 1}$

$= \frac{12 + 14}{20 + 7}$

$= \frac{26}{27}$

Question 3.

If a: b = 3: 8, find the value of $\frac{4a + 3b}{6a - b}$.

Solution:

$a : b = 3 : 8$

$\Rightarrow \frac{a}{b} = \frac{3}{8}$

$\frac{4a + 3b}{6a - b} = \frac{4\left(\frac{a}{b}\right) + 3}{6\left(\frac{a}{b}\right) - 1}$ (Dividing each term by b)

$= \frac{4\left(\frac{3}{8}\right) + 3}{6\left(\frac{3}{8}\right) - 1}$

$$= \frac{\frac{3}{2}+3}{\frac{9}{4}-1}$$

$$= \frac{\frac{9}{2}}{\frac{5}{4}}$$

$$= \frac{18}{5}$$

Question 4.

If (a − b): (a + b) = 1: 11, find the ratio (5a + 4b + 15): (5a − 4b + 3).

Solution:

$$\frac{a-b}{a+b} = \frac{1}{11}$$

$$11a - 11b = a + b$$

$$10a = 12b$$

$$\frac{a}{b} = \frac{12}{10} = \frac{6}{5}$$

So, let $a = 6k$ and $b = 5k$

$$\frac{5a+4b+15}{5a-4b+3} = \frac{5(6k)+4(5k)+15}{5(6k)-4(5k)+3}$$

$$= \frac{30k+20k+15}{30k-20k+3}$$

$$= \frac{50k+15}{10k+3}$$

$$= \frac{5(10k+3)}{10k+3}$$

$$= 5$$

Hence, (5a + 4b + 15): (5a - 4b + 3) = 5: 1

Question 5.

Find the number which bears the same ratio to $\frac{7}{33}$ that $\frac{8}{21}$ does to $\frac{4}{9}$.

Solution:

Let the required number be $\frac{x}{y}$.

Now, Ratio of $\frac{8}{21}$ to $\frac{4}{9} = \frac{8/21}{4/9} = \frac{8}{21} \times \frac{9}{4} = \frac{6}{7}$

Thus, we have

$$\frac{x/y}{7/33} = \frac{6}{7}$$

$$\Rightarrow \frac{x}{y} = \frac{6/7}{7/33}$$

$$\Rightarrow \frac{x}{y} = \frac{6}{7} \times \frac{7}{33}$$

$$\Rightarrow \frac{x}{y} = \frac{2}{11}$$

Hence, the required number is $\frac{2}{11}$.

Question 6.

If $\frac{m+n}{m+3n} = \frac{2}{3}$, find : $\frac{2n^2}{3m^2 + mn}$.

Solution:

$$\frac{m+n}{m+3n} = \frac{2}{3}$$

$$\Rightarrow 3m + 3n = 2m + 6n$$

$$\Rightarrow m = 3n$$

$$\Rightarrow \frac{m}{n} = \frac{3}{1}$$

$$\frac{2n^2}{3m^2 + mn} = \frac{2}{3\left(\frac{m}{n}\right)^2 + \left(\frac{m}{n}\right)} \text{ (Dividing each term by } n^2\text{)}$$

$$= \frac{2}{3\left(\frac{3}{1}\right)^2 + \left(\frac{3}{1}\right)}$$

$$= \frac{2}{27+3} = \frac{1}{15}$$

Question 7.

Find $\frac{x}{y}$, when $x^2 + 6y^2 = 5xy$.

Solution:

$x^2 + 6y^2 = 5xy$

Dividing both sides by y^2, we get,

$$\frac{x^2}{y^2} + \frac{6y^2}{y^2} = \frac{5xy}{y^2}$$

$$\left(\frac{x}{y}\right)^2 + 6 = 5\left(\frac{x}{y}\right)$$

$$\left(\frac{x}{y}\right)^2 - 5\left(\frac{x}{y}\right) + 6 = 0$$

Let $\frac{x}{y} = a$

$\therefore a^2 - 5a + 6 = 0$

$\Rightarrow (a-2)(a-3) = 0$

$\Rightarrow a = 2, 3$

Hence, $\frac{x}{y} = 2, 3$

Question 8.

If the ratio between 8 and 11 is the same as the ratio of 2x - y to x + 2y, find the value of $\frac{7x}{9y}$.

Solution:

Given,

$$\frac{2x-y}{x+2y}=\frac{8}{11}$$
$$22x-11y=8x+16y$$
$$14x=27y$$
$$\frac{x}{y}=\frac{27}{14}$$
$$\therefore \frac{7x}{9y}=\frac{7\times 27}{9\times 14}=\frac{3}{2}$$

Question 9.

Divide Rs 1,290 into A, B and C such that A is $\frac{2}{5}$ of B and B : C = 4 : 3.

Solution:

Given, $B:C=4:3 \Rightarrow \frac{B}{C}=\frac{4}{3}$

And, $A=\frac{2}{5}B \Rightarrow \frac{A}{B}=\frac{2}{5}$

Now, $\frac{A}{B}=\frac{2}{5}=\frac{2\times 4}{5\times 4}=\frac{8}{20}$ and $\frac{B}{C}=\frac{4\times 5}{3\times 5}=\frac{20}{15}$

$\Rightarrow A:B:C=8:20:15$

$\Rightarrow A=8x$, $B=20x$ and $C=15x$

$\therefore 8x+20x+15x=1290$

$\Rightarrow 43x=1290$

$\Rightarrow x=30$

A's share $=8x=8\times 30=$ Rs. 240

B's share $=20x=20\times 30=$ Rs. 600

C's share $=15x=15\times 30=$ Rs. 450

Question 10.
A school has 630 students. The ratio of the number of boys to the number of girls is 3 : 2. This ratio changes to 7 : 5 after the admission of 90 new students. Find the number of newly admitted boys.

Solution:

Let the number of boys be 3x.
Then, number of girls = 2x
$\therefore 3x + 2x = 630$
$\Rightarrow 5x = 630$
$\Rightarrow x = 126$
$\Rightarrow$ Number of boys = $3x = 3 \times 126 = 378$
And, Number of girls = $2x = 2 \times 126 = 252$

After admission of 90 new students, we have
total number of students = $630 + 90 = 720$
Now, let the number of boys be 7x.
Then, number of girls = 5x
$\therefore 7x + 5x = 720$
$\Rightarrow 12x = 720$
$\Rightarrow x = 60$
$\Rightarrow$ Number of boys = $7x = 7 \times 60 = 420$
And, Number of girls = $5x = 5 \times 60 = 300$

$\therefore$ Number of newly admitted boys = $420 - 378 = 42$

Question 11.
What quantity must be subtracted from each term of the ratio 9: 17 to make it equal to 1: 3?

Solution:

Let x be subtracted from each term of the ratio 9: 17.

$$\frac{9-x}{17-x}=\frac{1}{3}$$
$27-3x=17-x$
$10=2x$
$x=5$
Thus, the required number which should be subtracted is 5.

Question 12.
The monthly pocket money of Ravi and Sanjeev are in the ratio 5 : 7. Their expenditures are in the ratio 3 : 5. If each saves Rs. 80 every month, find their monthly pocket money.

Solution:

Given that the pocket money of Ravi and Sanjeev are in the ratio 5 : 7
Thus, the pocket money of Ravi is 5k and that of Sanjeev is 7k.
Also given that the expenditure of Ravi and Sanjeev are in the ratio 3 : 5
Thus, the expenditure of Ravi is 3m and that of Sanjeev is 5m.
And each of them saves Rs. 80
$\Rightarrow 5k-3m=80....(1)$
$7k-5m=80....(2)$
Solving equations (1) and (2), we have,
$k=40, m=40$
Hence the monthly pocket money of Ravi is Rs. 200 and that of Sanjeev is Rs. 280.

Question 13.
The work done by (x – 2) men in (4x + 1) days and the work done by (4x + 1) men in (2x – 3) days are in the ratio 3: 8. Find the value of x.

Solution:
Assuming that all the men do the same amount of work in one day and one day work of each man = 1 units, we have,

Amount of work done by (x – 2) men in (4x + 1) days
= Amount of work done by (x – 2)(4x + 1) men in one day
= (x – 2)(4x + 1) units of work

Similarly,

Amount of work done by (4x + 1) men in (2x – 3) days
= (4x + 1)(2x – 3) units of work

According to the given information,

$$\frac{(x-2)(4x+1)}{(4x+1)(2x-3)} = \frac{3}{8}$$
$$\frac{x-2}{2x-3} = \frac{3}{8}$$
$$8x - 16 = 6x - 9$$
$$2x = 7$$
$$x = \frac{7}{2} = 3.5$$

Question 14.
The bus fare between two cities is increased in the ratio 7: 9. Find the increase in the fare, if:
(i) the original fare is Rs 245;
(ii) the increased fare is Rs 207.

Solution:

According to the given information,
Increased (new) bus fare = $\frac{9}{7}$ x original bus fare
(i) We have:
Increased (new) bus fare = $\frac{9}{7}$ x Rs 245 = Rs 315
∴ Increase in fare = Rs 315 - Rs 245 = Rs 70
(ii) We have:
Rs 207 = $\frac{9}{7}$ x original bus fare
Original bus fare = Rs 207 x $\frac{7}{9}$ = Rs 161
∴ Increase in fare = Rs 207 - Rs 161 = Rs 46

Question 15.
By increasing the cost of entry ticket to a fair in the ratio 10: 13, the number of visitors to the fair has decreased in the ratio 6: 5. In what ratio has the total collection increased

or decreased?

Solution:
Let the cost of the entry ticket initially and at present be 10 x and 13x respectively.
Let the number of visitors initially and at present be 6y and 5y respectively.
Initially, total collection = 10x × 6y = 60 xy

At present, total collection = 13x × 5y = 65 xy
Ratio of total collection = 60 xy: 65 xy = 12: 13
Thus, the total collection has increased in the ratio 12: 13.

Question 16.
In a basket, the ratio between the number of oranges and the number of apples is 7: 13. If 8 oranges and 11 apples are eaten, the ratio between the number of oranges and the number of apples becomes 1: 2. Find the original number of oranges and the original number of apples in the basket.

Solution:
Let the original number of oranges and apples be 7x and 13x.
According to the given information,

$$\frac{7x-8}{13x-11}=\frac{1}{2}$$
$$14x-16=13x-11$$
$$x=5$$

Thus, the original number of oranges and apples are 7 × 5 = 35 and 13 × 5 = 65 respectively.

Question 17.
In a mixture of 126 kg of milk and water, milk and water are in ratio 5 : 2. How much water must be added to the mixture to make this ratio 3 : 2?

Solution:

Quantity of milk : Quantity of water = 5 : 2

$\therefore$ Quantity of milk $= 126 \times \frac{5}{7} = 90$ kg

$\Rightarrow$ Quantity of water = 126 − 90 = 36 kg

New ratio = 3 : 2

Let the quantity of water to be added be x kg.

Then, milk : water = $\frac{90}{36+x}$

$\therefore \frac{90}{36+x} = \frac{3}{2}$

$\Rightarrow 180 = 108 + 3x$

$\Rightarrow 3x = 72$

$\Rightarrow x = 24$

Thus, quantity of water to be added is 24 kg.

Question 18.

(A) If A: B = 3: 4 and B: C = 6: 7, find:

(i) A: B: C

(ii) A: C

(B) If A : B = 2 : 5 and A : C = 3 : 4, find

(i) A : B : C

Solution:

(A)

(i)

$\frac{A}{B} = \frac{3}{4} = \frac{3}{4} \times \frac{3}{3} = \frac{9}{12}$

$\frac{B}{C} = \frac{6}{7} = \frac{6}{7} \times \frac{2}{2} = \frac{12}{14}$

A : B : C = 9 : 12 : 14

(ii)

$\frac{A}{B} = \frac{3}{4}$

$\frac{B}{C} = \frac{6}{7}$

$\therefore \frac{A}{C} = \frac{\frac{A}{B}}{\frac{C}{B}} = \frac{\frac{3}{4}}{\frac{7}{6}} = \frac{3}{4} \times \frac{6}{7} = \frac{9}{14}$

$\therefore$ A : C = 9 : 14

(B)

(i)

To compare 3 ratios, the consequent of the first ratio and the antecedent of the 2nd ratio must be made equal.

Given that A : B = 2 : 5 and A : C = 3 : 4

Interchanging the first ratio, we have,

B : A = 5 : 2 and A : C = 3 : 4

L.C.M. of 2 and 3 is 6.

$\Rightarrow B : A = 5 \times 3 : 2 \times 3$ and $A : C = 3 \times 2 : 4 \times 2$

$\Rightarrow B : A = 15 : 6$ and $A : C = 6 : 8$

$\Rightarrow B : A : C = 15 : 6 : 8$

$\Rightarrow A : B : C = 6 : 15 : 8$

Question 19(i).

If 3A = 4B = 6C; find A: B: C.

Solution:

3A = 4B = 6C

$3A = 4B \Rightarrow \frac{A}{B} = \frac{4}{3}$

$4B = 6C \Rightarrow \frac{B}{C} = \frac{6}{4} = \frac{3}{2}$

Hence, A: B: C = 4: 3: 2

Question 19(ii).

If 2a = 3b and 4b = 5c, find: a : c.

Solution:

We have,

$2a = 3b \Rightarrow \frac{a}{b} = \frac{3}{2}$

And $4b = 5c \Rightarrow \frac{b}{c} = \frac{5}{4}$

Now, $\frac{a}{b} = \frac{3}{2} = \frac{3 \times 5}{2 \times 5} = \frac{15}{10}$ and $\frac{b}{c} = \frac{5}{4} = \frac{5 \times 2}{4 \times 2} = \frac{10}{8}$

$\Rightarrow a : b : c = 15 : 10 : 8$

$\Rightarrow a : c = 15 : 8$

Question 20.

Find the compound ratio of:

(i) 2: 3, 9: 14 and 14: 27

(ii) 2a: 3b, mn: x^2 and x: n.

(iii) $\sqrt{2} : 1, 3 : \sqrt{5}$ and $\sqrt{20} : 9$.

Solution:

(i) Required compound ratio = 3 × 8: 5 × 15

$= \frac{3 \times 8}{5 \times 15}$

$= \frac{8}{25} = 8 : 25$

(ii) Required compound ratio = 2 × 9 × 14: 3 × 14 × 27

$= \frac{2 \times 9 \times 14}{3 \times 14 \times 27}$

$= \frac{2}{9} = 2 : 9$

(iii) Required compound ratio = 2a × mn × x: 3b × x^2 × n

$= \frac{2a \times mn \times x}{3b \times x^2 \times n}$

$= \frac{2am}{3bx} = 2am : 3bx$

(iv) Required compound ratio = $\sqrt{2} \times 3 \times \sqrt{20} : 1 \times \sqrt{5} \times 9$

$= \frac{\sqrt{2} \times 3 \times \sqrt{20}}{1 \times \sqrt{5} \times 9}$

$= \frac{\sqrt{2} \times \sqrt{4}}{3}$

$= \frac{2\sqrt{2}}{3} = 2\sqrt{2} : 3$

Question 21.

Find duplicate ratio of:

(i) 3: 4 (ii) 3√3 : 2√5

Solution:

(i) Duplicate ratio of 3 : 4 = 32 : 42 = 9 : 16

(ii) Duplicate ratio of 3√3 : 2√5 = $(3\sqrt{3})^2 : (2\sqrt{5})^2$ = 27 : 20

Question 22.
Find the triplicate ratio of:
(i) 1: 3 (ii) $\frac{m}{2}:\frac{n}{3}$

Solution:

(i) Triplicate ratio of 1: 3 = $1^3 : 3^3$ = 1: 27
(ii) Triplicate ratio of $\frac{m}{2}:\frac{n}{3}$

$$= \left(\frac{m}{2}\right)^3 : \left(\frac{n}{3}\right)^3 = \frac{m^3}{8} : \frac{n^3}{27} = \frac{\frac{m^3}{8}}{\frac{n^3}{27}} = 27m^3 : 8n^3$$

Question 23.
Find sub-duplicate ratio of:
(i) 9: 16 (ii) $(x - y)^4 : (x + y)^6$

Solution:

(i) Sub-duplicate ratio of 9: 16 = $\sqrt{9} : \sqrt{16} = 3 : 4$
(ii) Sub-duplicate ratio of $(x - y)^4 : (x + y)^6$
$= \sqrt{(x-y)^4} : \sqrt{(x+y)^6} = (x-y)^2 : (x+y)^3$

Question 24.
Find the sub-triplicate ratio of:
(i) 64: 27 (ii) $x^3 : 125y^3$

Solution:
(i) Sub-triplicate ratio of 64 : 27 = $\sqrt[3]{64} : \sqrt[3]{27}$ = 4 : 3
(ii) Sub-triplicate ratio of $x^3 : 125y^3 = \sqrt[3]{x^3} : \sqrt[3]{125y^3} = x : 5y$

Question 25.
Find the reciprocal ratio of:
(i) 5: 8 (ii) $\frac{x}{3}:\frac{y}{7}$

Solution:

(i) Reciprocal ratio of 5: 8 = $\frac{1}{5} : \frac{1}{8} = 8 : 5$

(ii) Reciprocal ratio of $\frac{x}{3} : \frac{y}{7} = \frac{1}{\frac{x}{3}} : \frac{1}{\frac{y}{7}} = \frac{3}{x} : \frac{7}{y} = \frac{\frac{3}{x}}{\frac{7}{y}} = \frac{3y}{7x} = 3y : 7x$

Question 26.
If (x + 3) : (4x + 1) is the duplicate ratio of 3 : 5, find the value of x.

Solution:

If (x + 3) : (4x + 1) is the duplicate ratio of 3 : 5, find the value of x.

We have,

$\frac{x+3}{4x+1} = \frac{3^2}{5^2}$

$\Rightarrow \frac{x+3}{4x+1} = \frac{9}{25}$

$\Rightarrow 25x + 75 = 36x + 9$

$\Rightarrow 11x = 66$

$\Rightarrow x = 6$

Question 27.
If m: n is the duplicate ratio of m + x: n + x; show that x^2 = mn.

Solution:

$\frac{m}{n} = \frac{(m+x)^2}{(n+x)^2}$

$\frac{m}{n} = \frac{m^2 + x^2 + 2mx}{n^2 + x^2 + 2nx}$

$mn^2 + mx^2 + 2mnx = m^2n + nx^2 + 2mnx$

$x^2(m-n) = mn(m-n)$

$x^2 = mn$

Question 28.
If (3x – 9) : (5x + 4) is the triplicate ratio of 3 : 4, find the value of x.

Solution:

We have,

$$\frac{3x-9}{5x+4} = \frac{3^3}{4^3}$$

$$\Rightarrow \frac{3x-9}{5x+4} = \frac{27}{64}$$

$$\Rightarrow \frac{3(x-3)}{5x+4} = \frac{27}{64}$$

$$\Rightarrow \frac{x-3}{5x+4} = \frac{9}{64}$$

$$\Rightarrow 64x - 192 = 45x + 36$$

$$\Rightarrow 19x = 228$$

$$\Rightarrow x = 12$$

Question 29.
Find the ratio compounded of the reciprocal ratio of 15: 28, the sub-duplicate ratio of 36: 49 and the triplicate ratio of 5: 4.

Solution:

Reciprocal ratio of 15: 28 = 28: 15
Sub-duplicate ratio of 36: 49 = $\sqrt{36} : \sqrt{49} = 6 : 7$
Triplicate ratio of 5: 4 = $5^3 : 4^3$ = 125: 64
Required compounded ratio

$$= = \frac{28 \times 6 \times 125}{15 \times 7 \times 64} = \frac{25}{8} = 25 : 8$$

Question 30(a).
If $r^2 = pq$, show that p : q is the duplicate ratio of (p + r) : (q + r).

Solution:

Given, $r^2 = pq$

Duplicate ratio of $(p+r):(q+r) = (p+r)^2 : (q+r)^2$

$= (p^2 + r^2 + 2pr) : (q^2 + r^2 + 2qr)$

$= (p^2 + pq + 2pr) : (q^2 + pq + 2qr)$

$= p(p + q + 2r) : q(q + p + 2r)$

$= p : q$

Thus, p : q is the duplicate ratio of $(p+r):(q+r)$.

Question 30(b).

If $(p - x) : (q - x)$ be the duplicate ratio of p : q

then show that: $\frac{1}{p} + \frac{1}{q} = \frac{1}{x}$

Solution:

We have,

$$\frac{(p-x)}{(q-x)} = \frac{p^2}{q^2}$$

$\Rightarrow q^2(p - x) = p^2(q - x)$

$\Rightarrow pq^2 - q^2x = p^2q - p^2x$

$\Rightarrow p^2x - q^2x = p^2q - pq^2$

$\Rightarrow x(p^2 - q^2) = pq(p - q)$

$\Rightarrow x(p - q)(p + q) = pq(p - q)$

$$\Rightarrow x = \frac{pq}{p+q}$$

$$\Rightarrow \frac{p+q}{pq} = \frac{1}{x}$$

$$\Rightarrow \frac{p}{pq} + \frac{q}{pq} = \frac{1}{x}$$

$$\Rightarrow \frac{1}{q} + \frac{1}{p} = \frac{1}{x}$$

$$\Rightarrow \frac{1}{p} + \frac{1}{q} = \frac{1}{x}$$

Exercise 7B

Question 1.
Find the fourth proportional to:
(i) 1.5, 4.5 and 3.5 (ii) 3a, $6a^2$ and $2ab^2$

Solution:
(i) Let the fourth proportional to 1.5, 4.5 and 3.5 be x.
$\Rightarrow 1.5 : 4.5 = 3.5 : x$
$\Rightarrow 1.5 \times x = 3.5\ 4.5$
$\Rightarrow x = 10.5$

(ii) Let the fourth proportional to 3a, $6a^2$ and $2ab^2$ be x.
$\Rightarrow 3a : 6a^2 = 2ab^2 : x$
$\Rightarrow 3a \times x = 2ab^2\ 6a^2$
$\Rightarrow 3a \times x = 12a^3b^2$
$\Rightarrow x = 4a^2b^2$

Question 2.
Find the third proportional to:
(i) $2\frac{2}{3}$ and 4 (ii) a - b and $a^2 - b^2$

Solution:
(i) Let the third proportional to $2\frac{2}{3}$ and 4 be x.

$\Rightarrow 2\frac{2}{3}$, 4, x are in continued proportion.

$\Rightarrow 2\frac{2}{3} : 4 = 4 : x$

$$\Rightarrow \frac{\frac{8}{3}}{4} = \frac{4}{x}$$

$$\Rightarrow x = 16 \times \frac{3}{8} = 6$$

(ii) Let the third proportional to a - b and $a^2 - b^2$ be x.
$\Rightarrow$ a - b, $a^2 - b^2$, x are in continued proportion.
$\Rightarrow a - b : a^2 - b^2 = a^2 - b^2 : x$

$$\Rightarrow \frac{a-b}{a^2-b^2} = \frac{a^2-b^2}{x}$$

$$\Rightarrow x = \frac{(a^2-b^2)^2}{a-b}$$

(i) Let the third proportional to $2\frac{2}{3}$ and 4 be x.

$\Rightarrow 2\frac{2}{3}$, 4, x are in continued proportion.

$\Rightarrow 2\frac{2}{3} : 4 = 4 : x$

$$\Rightarrow \frac{\frac{8}{3}}{4} = \frac{4}{x}$$

$$\Rightarrow x = 16 \times \frac{3}{8} = 6$$

(ii) Let the third proportional to a - b and $a^2 - b^2$ be x.

$\Rightarrow$ a - b, $a^2 - b^2$, x are in continued proportion.

$\Rightarrow a - b : a^2 - b^2 = a^2 - b^2 : x$

$$\Rightarrow \frac{a-b}{a^2-b^2} = \frac{a^2-b^2}{x}$$

$$\Rightarrow x = \frac{(a^2-b^2)^2}{a-b}$$

$$\Rightarrow x = \frac{(a+b)(a-b)(a^2-b^2)}{a-b}$$

$$\Rightarrow x = (a+b)(a^2-b^2)$$

Question 3.

Find the mean proportional between:

(i) $6 + 3\sqrt{3}$ and $8 - 4\sqrt{3}$

(ii) a - b and $a^3 - a^2b$

Solution:

(i) Let the mean proportional between $6 + 3\sqrt{3}$ and $8 - 4\sqrt{3}$ be x.

$\Rightarrow 6 + 3\sqrt{3}$, x and $8 - 4\sqrt{3}$ are in continued proportion.

$\Rightarrow 6 + 3\sqrt{3} : x = x : 8 - 4\sqrt{3}$

$\Rightarrow x \times x = (6 + 3\sqrt{3})(8 - 4\sqrt{3})$

$\Rightarrow x^2 = 48 + 24\sqrt{3} - 24\sqrt{3} - 36$

$\Rightarrow x^2 = 12$

$\Rightarrow x = 2\sqrt{3}$

(ii) Let the mean proportional between $a - b$ and $a^3 - a^2b$ be x.

$\Rightarrow a - b$, x, $a^3 - a^2b$ are in continued proportion.

$\Rightarrow a - b : x = x : a^3 - a^2b$

$\Rightarrow x \times x = (a - b)(a^3 - a^2b)$

$\Rightarrow x^2 = (a - b)\, a^2(a - b) = [a(a - b)]^2$
$\Rightarrow x = a(a - b)$

Question 4.
If x + 5 is the mean proportional between x + 2 and x + 9; find the value of x.

Solution:
Given, x + 5 is the mean proportional between x + 2 and x + 9.
$\Rightarrow (x + 2), (x + 5)$ and $(x + 9)$ are in continued proportion.
$\Rightarrow (x + 2) : (x + 5) = (x + 5) : (x + 9)$
$\Rightarrow (x + 5)^2 = (x + 2)(x + 9)$
$\Rightarrow x^2 + 25 + 10x = x^2 + 2x + 9x + 18$
$\Rightarrow 25 - 18 = 11x - 10x$
$\Rightarrow x = 7$

Question 5.
If x^2, 4 and 9 are in continued proportion, find x.

Solution:
Given, x^2, 4 and 9 are in continued proportion.

$\therefore \frac{x^2}{4} = \frac{4}{9}$

$\Rightarrow 9x^2 = 16$

$\Rightarrow x^2 = \frac{16}{9}$

$\Rightarrow x = \frac{4}{3}$

Question 6.
What least number must be added to each of the numbers 6, 15, 20 and 43 to make them proportional?

Solution:

Let the number added be x.
$\therefore (6 + x) : (15 + x) :: (20 + x) (43 + x)$

$\frac{6 + x}{15 + x} = \frac{20 + x}{43 + x}$

$(6 + x)(43 + x) = (20 + x)(15 + x)$

$$258 + 6x + 43x + x^2 = 300 + 20x + 15x + x^2$$
$$49x - 35x = 300 - 258$$
$$14x = 42$$
$$x = 3$$

Thus, the required number which should be added is 3.

Question 7(i).

If a, b, c are in continued proportion,

show that: $\frac{a^2 + b^2}{b(a+c)} = \frac{b(a+c)}{b^2 + c^2}$.

Solution:

Since a, b, c are in continued proportion,

$$\frac{a}{b} = \frac{b}{c}$$
$$\Rightarrow b^2 = ac$$

$$\text{Now, } (a^2 + b^2)(b^2 + c^2) = (a^2 + ac)(ac + c^2)$$
$$= a(a + c)c(a + c)$$
$$= ac(a + c)^2$$
$$= b^2(a + c)^2$$
$$\Rightarrow (a^2 + b^2)(b^2 + c^2) = [b(a + c)][b(a + c)]$$
$$\Rightarrow \frac{a^2 + b^2}{b(a + c)} = \frac{b(a + c)}{b^2 + c^2}$$

Question 7(ii).

If a, b, c are in continued proportion and $a(b - c) = 2b$,

prove that : $a - c = \frac{2(a + b)}{a}$.

Solution:

Since a, b, c are in continued proportion,

$\frac{a}{b} = \frac{b}{c}$

$\Rightarrow b^2 = ac$

$a(b - c) = 2b$

$\Rightarrow ab - ac = 2b$

$\Rightarrow ab - b^2 = 2b$

$\Rightarrow b(a - b) = 2b$

$\Rightarrow a - b = 2$

Now,

L.H.S. $= a - c$

$= \frac{a(a - c)}{a}$

$= \frac{a^2 - ac}{a}$

$= \frac{a^2 - b^2}{a}$

$= \frac{(a - b)(a + b)}{a}$

$= \frac{2(a + b)}{2}$

= R.H.S.

Question 7(iii).

If $\frac{a}{b} = \frac{c}{d}$, show that: $\frac{a^3c + ac^3}{b^3d + bd^3} = \frac{(a + c)^4}{(b + d)^4}$.

Solution:

Let $\frac{a}{b} = \frac{c}{d} = k$

$\Rightarrow a = bk$ and $c = dk$

L.H.S. $= \frac{a^3c + ac^3}{b^3d + bd^3}$

$$= \frac{ac(a^2 + c^2)}{bd(b^2 + d^2)}$$

$$= \frac{(bk \times dk)(b^2k^2 + d^2k^2)}{bd(b^2 + d^2)}$$

$$= \frac{k^2 \times k^2(b^2 + d^2)}{(b^2 + d^2)}$$

$$= k^4$$

$$\text{R.H.S.} = \frac{(a+c)^4}{(b+d)^4} = \frac{(bk + dk)^4}{(b+d)^4} = \left[\frac{k(b+d)}{b+d}\right]^4 = k^4$$

Hence, $\frac{a^3c + ac^3}{b^3d + bd^3} = \frac{(a+c)^4}{(b+d)^4}$

Question 8.
What least number must be subtracted from each of the numbers 7, 17 and 47 so that the remainders are in continued proportion?

Solution:

Let the number subtracted be x.

$\therefore$ $(7 - x) : (17 - x) :: (17 - x) (47 - x)$

$$\frac{7 - x}{17 - x} = \frac{17 - x}{47 - x}$$

$$(7 - x)(47 - x) = (17 - x)^2$$

$$329 - 47x - 7x + x^2 = 289 - 34x + x^2$$

$$329 - 289 = -34x + 54x$$

$$20x = 40$$

$$x = 2$$

Thus, the required number which should be subtracted is 2.

Question 9.
If y is the mean proportional between x and z; show that xy + yz is the mean proportional between x^2+y^2 and y^2+z^2.

Solution:
Since y is the mean proportion between x and z
Therefore, $y^2 = xz$

Now, we have to prove that xy+yz is the mean proportional between x^2+y^2 and y^2+z^2, i.e.,

$(xy + yz)^2 = (x^2 + y^2)(y^2 + z^2)$

$LHS = (xy + yz)^2$
$= [y(x + z)]^2$
$= y^2(x + z)^2$
$= xz(x + z)^2$
$RHS = (x^2 + y^2)(y^2 + z^2)$
$= (x^2 + xz)(xz + z^2)$
$= x(x + z)z(x + z)$
$= xz(x + z)^2$
LHS = RHS
Hence, proved.

Question 10.
If q is the mean proportional between p and r, show that:
$pqr (p + q + r)^3 = (pq + qr + rp)^3$.

Solution:
Given, q is the mean proportional between p and r.

$\Rightarrow q^2 = pr$

$$
\begin{aligned}
L.H.S. &= pqr(p + q + r)^3 \\
&= qq^2(p + q + r)^3 && [\because q^2 = pr] \\
&= q^3(p + q + r)^3 \\
&= [q(p + q + r)]^3 \\
&= (pq + q^2 + qr)^3 \\
&= (pq + pr + qr)^3 && [\because q^2 = pr] \\
&= R.H.S.
\end{aligned}
$$

Question 11.
If three quantities are in continued proportion; show that the ratio of the first to the third is the duplicate ratio of the first to the second.

Solution:
Let x, y and z be the three quantities which are in continued proportion.
Then, $x : y :: y : z \Rightarrow y^2 = xz$ ....(1)

Now, we have to prove that
$x : z = x^2 : y^2$
That is we need to prove that
$xy^2 = x^2z$
LHS = $xy^2 = x(xz) = x^2z$ = RHS [Using (1)]
Hence, proved.

Question 12.

If y is the mean proportional between x and z, prove that:

$$\frac{x^2 - y^2 + z^2}{x^{-2} - y^{-2} + z^{-2}} = y^4.$$

Solution:
Given, y is the mean proportional between x and z.

$\Rightarrow y^2 = xz$

$$\text{LHS} = \frac{x^2 - y^2 + z^2}{x^{-2} - y^{-2} + z^{-2}}$$

$$= \frac{x^2 - y^2 + z^2}{\frac{1}{x^2} - \frac{1}{y^2} + \frac{1}{z^2}}$$

$$= \frac{x^2 - xz + z^2}{\frac{1}{x^2} - \frac{1}{xz} + \frac{1}{z^2}} \qquad (\because y^2 = xz)$$

$$= \frac{x^2 - xz + z^2}{\frac{z^2 - xz + x^2}{x^2z^2}}$$

$$= x^2z^2$$

$$= (xz)^2$$

$$= (y^2)^2 \qquad (\because y^2 = xz)$$

$$= y^4$$

$$= \text{RHS}$$

Question 13.

Given four quantities a, b, c and d are in proportion. Show that:

$(a - c)b^2 : (b - d)cd = (a^2 - b^2 - ab) : (c^2 - d^2 - cd)$

Solution:

Let $\frac{a}{b} = \frac{c}{d} = k$

$\Rightarrow a = bk \text{ and } c = dk$

$$\text{LHS} = \frac{(a-c)b^2}{(b-d)cd}$$

$$= \frac{(bk-dk)b^2}{(b-d)dkd}$$

$$= \frac{k(b-d)b^2}{(b-d)d^2k}$$

$$= \frac{b^2}{d^2}$$

$$\text{RHS} = \frac{(a^2-b^2-ab)}{(c^2-d^2-cd)}$$

$$= \frac{(b^2k^2-b^2-bkb)}{(d^2k^2-d^2-dkd)}$$

$$= \frac{b^2(k^2-1-k)}{d^2(k^2-1-k)}$$

$$= \frac{b^2}{d^2}$$

$\Rightarrow$ LHS = RHS

Hence proved.

Question 14.
Find two numbers such that the mean mean proportional between them is 12 and the third proportional to them is 96.

Solution:

Let a and b be the two numbers, whose mean proportional is 12.

$\therefore\ ab = 12^2 \Rightarrow ab = 144 \Rightarrow b = \dfrac{144}{a}$ ....(i)

Now, third proportional is 96

$\therefore\ a : b :: b : 96$

$\Rightarrow b^2 = 96a$

$\Rightarrow \left(\dfrac{144}{a}\right)^2 = 96a$

$\Rightarrow \dfrac{(144)^2}{a^2} = 96a$

$\Rightarrow a^3 = \dfrac{144 \times 144}{96}$

$\Rightarrow a^3 = 216$

$\Rightarrow a = 6$

$b = \dfrac{144}{6} = 24$

Therefore, the numbers are 6 and 24.

Question 15.

Find the third proportional to $\frac{x}{y} + \frac{y}{x}$ and $\sqrt{x^2 + y^2}$

Solution:

Let the required third proportional be p.

$\Rightarrow \dfrac{x}{y} + \dfrac{y}{x}, \sqrt{x^2 + y^2}$, p are in continued proportion.

$\Rightarrow \dfrac{x}{y} + \dfrac{y}{x} : \sqrt{x^2 + y^2} = \sqrt{x^2 + y^2} : p$

$\Rightarrow p\left(\dfrac{x}{y} + \dfrac{y}{x}\right) = \left(\sqrt{x^2 + y^2}\right)^2$

$\Rightarrow p\left(\dfrac{x^2 + y^2}{xy}\right) = x^2 + y^2$

$\Rightarrow p = xy$

Question 16.
If p: q = r: s; then show that:
mp + nq : q = mr + ns : s.

Solution:

$$\frac{p}{q} = \frac{r}{s}$$
$$\Rightarrow \frac{mp}{q} = \frac{mr}{s}$$
$$\Rightarrow \frac{mp}{q} + n = \frac{mr}{s} + n$$
$$\Rightarrow \frac{mp + nq}{q} = \frac{mr + ns}{s}$$

Hence, mp + nq : q = mr + ns : s.

Question 17.

If p + r = mq and $\frac{1}{q} + \frac{1}{s} = \frac{m}{r}$; then prove that p : q = r : s.

Solution:

$$\frac{1}{q} + \frac{1}{s} = \frac{m}{r}$$
$$\frac{s + q}{qs} = \frac{m}{r}$$
$$\frac{s + q}{s} = \frac{mq}{r}$$
$$\frac{s + q}{s} = \frac{p + r}{r} \quad (\because p + r = mq)$$
$$1 + \frac{q}{s} = \frac{p}{r} + 1$$
$$\frac{q}{s} = \frac{p}{r}$$
$$\frac{p}{q} = \frac{r}{s}$$

Hence, proved.

Question 18.

If $\frac{a}{b} = \frac{c}{d}$, prove that each of the given ratio is equal to:

(i) $\frac{5a + 4c}{5b + 4d}$

(ii) $\frac{13a - 8c}{13b - 8d}$

(iii) $\sqrt{\frac{3a^2 - 10c^2}{3b^2 - 10d^2}}$

(iv) $\left(\frac{8a^3 + 15c^3}{8b^3 + 15d^3}\right)^{\frac{1}{3}}$

Solution:

Let $\frac{a}{b} = \frac{c}{d} = k$

Then, a = bk and c = dk

(i) $\frac{5a + 4c}{5b + 4d} = \frac{5(bk) + 4(dk)}{5b + 4d} = \frac{k(5b + 4d)}{5b + 4d} = k =$ each given ratio

(ii) $\frac{13a - 8c}{13b - 8d} = \frac{13(bk) - 8(dk)}{13b - 8d} = \frac{k(13b - 8d)}{13b - 8d} = k =$ each given ratio

(iii) $\sqrt{\frac{3a^2 - 10c^2}{3b^2 - 10d^2}} = \sqrt{\frac{3(bk)^2 - 10(dk)^2}{3b^2 - 10d^2}} = \sqrt{\frac{k^2(3b^2 - 10d^2)}{3b^2 - 10d^2}} = k$

= each given ratio

(iv) $\left(\frac{8a^3 + 15c^3}{8b^3 + 15d^3}\right)^{\frac{1}{3}} = \left[\frac{8(bk)^3 + 15(dk)^3}{8b^3 + 15d^3}\right]^{\frac{1}{3}} = \left[\frac{k^3(8b^3 + 15d^3)}{8b^3 + 15d^3}\right]^{\frac{1}{3}} = k$

= each given ratio

Question 19.

If a, b, c and d are in proportion, prove that:

(i) $\frac{13a+17b}{13c+17d} = \sqrt{\frac{2ma^2 - 3nb^2}{2mc^2 - 3nd^2}}$

(ii) $\sqrt{\frac{4a^2 + 9b^2}{4c^2 + 9d^2}} = \left(\frac{xa^3 - 5yb^3}{xc^3 - 5yd^3}\right)^{\frac{1}{3}}$

Solution:

a, b, c and d are in proportion

$\frac{a}{b} = \frac{c}{d} = k \text{ (say)}$

Then, a = bk and c = dk

(i) L.H.S. $= \frac{13a+17b}{13c+17d} = \frac{13(bk)+17b}{13(dk)+17d} = \frac{b(13k+17)}{d(13k+17)} = \frac{b}{d}$

R.H.S. $= \sqrt{\frac{2ma^2 - 3nb^2}{2mc^2 - 3nd^2}} = \sqrt{\frac{2m(bk)^2 - 3nb^2}{2m(dk)^2 - 3nd^2}} = \sqrt{\frac{b^2(2mk^2 - 3n)}{d^2(2mk^2 - 3n)}} = \frac{b}{d}$

Hence, L.H.S. = R.H.S.

(ii) L.H.S. $= \sqrt{\frac{4a^2 + 9b^2}{4c^2 + 9d^2}} = \sqrt{\frac{4(bk)^2 + 9b^2}{4(dk)^2 + 9d^2}} = \sqrt{\frac{b^2(4k^2 + 9)}{d^2(4k^2 + 9)}} = \frac{b}{d}$

R.H.S. $= \left(\frac{xa^3 - 5yb^3}{xc^3 - 5yd^3}\right)^{\frac{1}{3}} = \left[\frac{x(bk)^3 - 5yb^3}{x(dk)^3 - 5yd^3}\right]^{\frac{1}{3}}$

$= \left[\frac{b^3(xk^3 - 5y)}{d^3(xk^3 - 5y)}\right]^{\frac{1}{3}}$

$= \left[\frac{b^3}{d^3}\right]^{\frac{1}{3}} = \frac{b}{d}$

Hence, L.H.S. = R.H.S.

Question 20.

If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove that:

$$\frac{2x^3 - 3y^3 + 4z^3}{2a^3 - 3b^3 + 4c^3} = \left(\frac{2x - 3y + 4z}{2a - 3b + 4c}\right)^3$$

Solution:

Let $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k$

Then, x = ak, y = bk and z = ck

$$\text{L.H.S.} = \frac{2x^3 - 3y^3 + 4z^3}{2a^3 - 3b^3 + 4c^3}$$
$$= \frac{2(ak)^3 - 3(bk)^3 + 4(ck)^3}{2a^3 - 3b^3 + 4c^3}$$
$$= \frac{2a^3k^3 - 3b^3k^3 + 4c^3k^3}{2a^3 - 3b^3 + 4c^3}$$
$$= \frac{k^3(2a^3 - 3b^3 + 4c^3)}{2a^3 - 3b^3 + 4c^3}$$
$$= k^3$$

$$\text{R.H.S.} = \left(\frac{2x - 3y + 4z}{2a - 3b + 4c}\right)^3$$
$$= \left(\frac{2ak - 3bk + 4ck}{2a - 3b + 4c}\right)^3$$
$$= \left[\frac{k(2a - 3b + 4c)}{2a - 3b + 4c}\right]^3$$
$$= k^3$$

Hence, L.H.S. = R.H.S.

Exercise 7C

Question 1.

If a : b = c : d, prove that:

(i) 5a + 7b : 5a – 7b = 5c + 7d : 5c – 7d.

(ii) (9a + 13b) (9c – 13d) = (9c + 13d) (9a – 13b).

(iii) xa + yb : xc + yd = b : d.

Solution:

(i)Given, $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \frac{5a}{7b} = \frac{5c}{7d}$ $\left(\text{Mutiplying each side by } \frac{5}{7}\right)$

$\Rightarrow \frac{5a + 7b}{5a - 7b} = \frac{5c + 7d}{5c - 7d}$ (By componendo and dividendo)

(ii)Given, $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \frac{9a}{13b} = \frac{9c}{13d}$ $\left(\text{Mutiplying each side by } \frac{9}{13}\right)$

$\Rightarrow \frac{9a + 13b}{13a - 13b} = \frac{9c + 13d}{9c - 13d}$ (By componendo and dividendo)

$\Rightarrow (9a + 13b)(9c - 13d) = (9c + 13d)(9a - 13b)$

(iii)Given, $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \frac{xa}{yb} = \frac{xc}{yd}$ $\left(\text{Mutiplying each side by } \frac{x}{y}\right)$

$\Rightarrow \frac{xa + yb}{yb} = \frac{xc + yd}{yd}$ (By componendo)

$\Rightarrow \frac{xa + yb}{xc + yd} = \frac{yb}{yd}$

$\Rightarrow \frac{xa + yb}{xc + yd} = \frac{b}{d}$

Question 2.
If a : b = c : d, prove that:
(6a + 7b) (3c – 4d) = (6c + 7d) (3a – 4b).

Solution:

Given, $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \frac{6a}{7b} = \frac{6c}{7d}$ $\left(\text{Mutiplying each side by } \frac{6}{7}\right)$

$\Rightarrow \frac{6a+7b}{7b} = \frac{6c+7d}{7d}$ (By componendo)

$\Rightarrow \frac{6a+7b}{6c+7d} = \frac{7b}{7d} = \frac{b}{d}$ ...(1)

Also, $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \frac{3a}{4b} = \frac{3c}{4d}$ $\left(\text{Mutiplying each side by } \frac{3}{4}\right)$

$\Rightarrow \frac{3a-4b}{4b} = \frac{3c-4d}{4d}$ (By dividendo)

$\Rightarrow \frac{3a-4b}{3c-4d} = \frac{4b}{4d} = \frac{b}{d}$ ...(2)

From (1) and (2),

$\frac{6a+7b}{6c+7d} = \frac{3a-4b}{3c-4d}$

$(6a+7b)(3c-4d) = (6c+7d)(3a-4b)$

Question 3.

Given, $\frac{a}{b} = \frac{c}{d}$, prove that:

$\frac{3a-5b}{3a+5b} = \frac{3c-5d}{3c+5d}$

Solution:

$\frac{a}{b} = \frac{c}{d}$

$\frac{3a}{5b} = \frac{3c}{5d}$ $\left(\text{Multiplying each side by } \frac{3}{5}\right)$

$\frac{3a+5b}{3a-5b} = \frac{3c+5d}{3c-5d}$ (By componendo and dividendo)

$\frac{3a-5b}{3a+5b} = \frac{3c-5d}{3c+5d}$ (By alternendo)

Question 4.

If $\frac{5x+6y}{5u+6v}=\frac{5x-6y}{5u-6v}$; then prove that:

x: y = u: v.

Solution:

$$\frac{5x+6y}{5u+6v}=\frac{5x-6y}{5u-6v}$$

(By alternendo)

$$\frac{5x+6y}{5x-6y}=\frac{5u+6v}{5u-6v}$$

$$\frac{5x+6y+5x-6y}{5x+6y-5x+6y}=\frac{5u+6v+5u-6v}{5u+6v-5u+6v}$$

(By componendo and dividendo)

$$\frac{10x}{12y}=\frac{10u}{12v}$$

$$\frac{x}{y}=\frac{u}{v}$$

Question 5.

If (7a + 8b) (7c – 8d) = (7a – 8b) (7c + 8d), prove that a: b = c: d.

Solution:

Given, $\frac{7a+8b}{7a-8b}=\frac{7c+8d}{7c-8d}$

Applying componendo and dividendo,

$$\frac{7a+8b+7a-8b}{7a+8b-7a+8b}=\frac{7c+8d+7c-8d}{7c+8d-7c+8d}$$

$$\Rightarrow \quad \frac{14a}{16b}=\frac{14c}{16d}$$

$$\Rightarrow \quad \frac{a}{b}=\frac{c}{d}$$

Hence, a: b = c: d.

Question 6.

(i) If $x = \frac{6ab}{a+b}$, find the value of:

$\frac{x+3a}{x-3a} + \frac{x+3b}{x-3b}$.

(ii) If $a = \frac{4\sqrt{6}}{\sqrt{2}+\sqrt{3}}$, find the value of:

$\frac{a+2\sqrt{2}}{a-2\sqrt{2}} + \frac{a+2\sqrt{3}}{a-2\sqrt{3}}$.

Solution:

(i) $x = \frac{6ab}{a+b}$

$\Rightarrow \frac{x}{3a} = \frac{2b}{a+b}$

Applying componendo and dividendo,

$\frac{x+3a}{x-3a} = \frac{2b+a+b}{2b-a-b}$

$\frac{x+3a}{x-3a} = \frac{3b+a}{b-a} \quad \ldots(1)$

Again, $x = \frac{6ab}{a+b}$

$\Rightarrow \frac{x}{3b} = \frac{2a}{a+b}$

Applying componendo and dividendo,

$\frac{x+3b}{x-3b} = \frac{2a+a+b}{2a-a-b}$

$\frac{x+3b}{x-3b} = \frac{3a+b}{a-b} \quad \ldots(2)$

From (1) and (2),

$\frac{x+3a}{x-3a} + \frac{x+3b}{x-3b} = \frac{3b+a}{b-a} + \frac{3a+b}{a-b}$

$\frac{x+3a}{x-3a} + \frac{x+3b}{x-3b} = \frac{-3b-a+3a+b}{a-b}$

$\frac{x+3a}{x-3a} + \frac{x+3b}{x-3b} = \frac{2a-2b}{a-b} = 2$

(ii) $a = \frac{4\sqrt{6}}{\sqrt{2}+\sqrt{3}}$

$$\frac{a}{2\sqrt{2}} = \frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}$$

Applying componendo and dividendo,

$$\frac{a+2\sqrt{2}}{a-2\sqrt{2}} = \frac{2\sqrt{3}+\sqrt{2}+\sqrt{3}}{2\sqrt{3}-\sqrt{2}-\sqrt{3}}$$

$$\frac{a+2\sqrt{2}}{a-2\sqrt{2}} = \frac{3\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \qquad \ldots(1)$$

$$\frac{a}{2\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{2}+\sqrt{3}}$$

Applying componendo and dividendo,

$$\frac{a+2\sqrt{3}}{a-2\sqrt{3}} = \frac{2\sqrt{2}+\sqrt{2}+\sqrt{3}}{2\sqrt{2}-\sqrt{2}-\sqrt{3}}$$

$$\frac{a+2\sqrt{3}}{a-2\sqrt{3}} = \frac{3\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}} \qquad \ldots(2)$$

From (1) and (2),

$$\frac{a+2\sqrt{2}}{a-2\sqrt{2}} + \frac{a+2\sqrt{3}}{a-2\sqrt{3}} = \frac{3\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} + \frac{3\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$$

$$\frac{a+2\sqrt{2}}{a-2\sqrt{2}} + \frac{a+2\sqrt{3}}{a-2\sqrt{3}} = \frac{3\sqrt{2}+\sqrt{3}-3\sqrt{3}-\sqrt{2}}{\sqrt{2}-\sqrt{3}}$$

$$\frac{a+2\sqrt{2}}{a-2\sqrt{2}} + \frac{a+2\sqrt{3}}{a-2\sqrt{3}} = \frac{2\sqrt{2}-2\sqrt{3}}{\sqrt{2}-\sqrt{3}} = 2$$

Question 7.
If (a + b + c + d) (a − b − c + d) = (a + b − c − d) (a − b + c − d), prove that a: b = c: d.

Solution:

Given, $\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}$

Applying componendo and dividendo,

$$\frac{(a+b+c+d)+(a+b-c-d)}{(a+b+c+d)-(a+b-c-d)}=\frac{(a-b+c-d)+(a-b-c+d)}{(a-b+c-d)-(a-b-c+d)}$$

$$\frac{2(a+b)}{2(c+d)}=\frac{2(a-b)}{2(c-d)}$$

$$\frac{a+b}{c+d}=\frac{a-b}{c-d}$$

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}$$

Applying componendo and dividendo,

$$\frac{a+b+a-b}{a+b-a+b}=\frac{c+d+c-d}{c+d-c+d}$$

$$\frac{2a}{2b}=\frac{2c}{2d}$$

$$\frac{a}{b}=\frac{c}{d}$$

Question 8.

If $\frac{a-2b-3c+4d}{a+2b-3c-4d}=\frac{a-2b+3c-4d}{a+2b+3c+4d}$, show that 2ad = 3bc.

Solution:

$$\frac{a-2b-3c+4d}{a+2b-3c-4d}=\frac{a-2b+3c-4d}{a+2b+3c+4d}$$

Applying componendo and dividendo,

$$\frac{(a-2b-3c+4d)+(a+2b-3c-4d)}{(a-2b-3c+4d)-(a+2b-3c-4d)}$$

$$=\frac{(a-2b+3c-4d)+(a+2b+3c+4d)}{(a-2b+3c-4d)-(a+2b+3c+4d)}$$

$$\frac{2(a-3c)}{2(-2b+4d)}=\frac{2(a+3c)}{2(-2b-4d)}$$

$$\frac{a-2b-3c+4d}{a+2b-3c-4d}=\frac{a-2b+3c-4d}{a+2b+3c+4d}$$

Applying componendo and dividendo,

$$\frac{(a-2b-3c+4d)+(a+2b-3c-4d)}{(a-2b-3c+4d)-(a+2b-3c-4d)}$$
$$=\frac{(a-2b+3c-4d)+(a+2b+3c+4d)}{(a-2b+3c-4d)-(a+2b+3c+4d)}$$

$$\frac{2(a-3c)}{2(-2b+4d)}=\frac{2(a+3c)}{2(-2b-4d)}$$

$$\frac{a-3c}{a+3c}=\frac{-2b+4d}{-2b-4d}$$

Applying componendo and dividendo,

$$\frac{a-3c+a+3c}{a-3c-a-3c}=\frac{-2b+4d-2b-4d}{-2b+4d+2b+4d}$$

$$\frac{2a}{-6c}=\frac{-4b}{8d}$$

$$\frac{a}{-3c}=\frac{-b}{2d}$$

$$2ad=3bc$$

Question 9.

If $(a^2+b^2)(x^2+y^2)=(ax+by)^2$; prove that: $\frac{a}{x}=\frac{b}{y}$.

Solution:

Given, $(a^2+b^2)(x^2+y^2)=(ax+by)^2$

$$a^2x^2+a^2y^2+b^2x^2+b^2y^2=a^2x^2+b^2y^2+2abxy$$

$$a^2y^2+b^2x^2-2abxy=0$$

$$(ay-bx)^2=0$$

$$ay-bx=0$$

$$ay=bx$$

$$\frac{a}{x}=\frac{b}{y}$$

Question 10.

If a, b and c are in continued proportion, prove that:

(i) $\frac{a^2+ab+b^2}{b^2+bc+c^2}=\frac{a}{c}$

(ii) $\frac{a^2+b^2+c^2}{(a+b+c)^2}=\frac{a-b+c}{a+b+c}$

Solution:

Given, a, b and c are in continued proportion.

$$\Rightarrow \frac{a}{b}=\frac{b}{c}=k \text{ (say)}$$

$$\Rightarrow a=bk, b=ck$$

$$\Rightarrow a=(ck)k=ck^2, b=ck$$

(i) $\text{L.H.S.}=\frac{a^2+ab+b^2}{b^2+bc+c^2}$

$$=\frac{(ck^2)^2+(ck^2)(ck)+(ck)^2}{(ck)^2+(ck)c+c^2}$$

$$=\frac{c^2k^4+c^2k^3+c^2k^2}{c^2k^2+c^2k+c^2}$$

$$=\frac{c^2k^2(k^2+k+1)}{c^2(k^2+k+1)}$$

$$=k^2$$

$\text{R.H.S.}=\frac{a}{c}=\frac{ck^2}{c}=k^2$

$\therefore$ L.H.S. = R.H.S.

(ii) L.H.S. $= \frac{a^2 + b^2 + c^2}{(a+b+c)^2}$

$= \frac{(ck^2)^2 + (ck)^2 + c^2}{(ck^2 + ck + c)^2}$

$= \frac{c^2k^4 + c^2k^2 + c^2}{c^2(k^2 + k + 1)^2}$

$= \frac{c^2(k^4 + k^2 + 1)}{c^2(k^2 + k + 1)^2}$

$= \frac{k^4 + k^2 + 1}{(k^2 + k + 1)^2}$

R.H.S. $= \frac{a - b + c}{a + b + c}$

$= \frac{ck^2 - ck + c}{ck^2 + ck + c}$

$= \frac{k^2 - k + 1}{k^2 + k + 1}$

$= \frac{(k^2 - k + 1)(k^2 + k + 1)}{(k^2 + k + 1)^2}$

$= \frac{k^4 + k^3 + k^2 - k^3 - k^2 - k + k^2 + k + 1}{(k^2 + k + 1)^2}$

$= \frac{k^4 + k^2 + 1}{(k^2 + k + 1)^2}$

$\therefore$ L.H.S. = R.H.S.

Question 11.

Using properties of proportion, solve for x:

(i) $\frac{\sqrt{x+5} + \sqrt{x-16}}{\sqrt{x+5} - \sqrt{x-16}} = \frac{7}{3}$

(ii) $\frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} = \frac{4x-1}{2}$

(iii) $\frac{3x + \sqrt{9x^2 - 5}}{3x - \sqrt{9x^2 - 5}} = 5$

Solution:

(i) $\frac{\sqrt{x+5}+\sqrt{x-16}}{\sqrt{x+5}-\sqrt{x-16}}=\frac{7}{3}$

Applying componendo and dividendo,

$$\frac{\sqrt{x+5}+\sqrt{x-16}+\sqrt{x+5}-\sqrt{x-16}}{\sqrt{x+5}+\sqrt{x-16}-\sqrt{x+5}+\sqrt{x-16}}=\frac{7+3}{7-3}$$

$$\frac{2\sqrt{x+5}}{2\sqrt{x-16}}=\frac{10}{4}$$

$$\frac{\sqrt{x+5}}{\sqrt{x-16}}=\frac{5}{2}$$

Squaring both sides,

$$\frac{x+5}{x-16}=\frac{25}{4}$$

$$4x+20=25x-400$$

$$21x=420$$

$$x=\frac{420}{21}=20$$

(ii) $\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}=\frac{4x-1}{2}$

Applying componendo and dividendo,

$$\frac{\sqrt{x+1}+\sqrt{x-1}+\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}-\sqrt{x+1}+\sqrt{x-1}}=\frac{4x-1+2}{4x-1-2}$$

$$\frac{2\sqrt{x+1}}{2\sqrt{x-1}}=\frac{4x+1}{4x-3}$$

Squaring both sides,

$$\frac{x+1}{x-1}=\frac{16x^2+1+8x}{16x^2+9-24x}$$

Applying componendo and dividendo,

$$\frac{x+1+x-1}{x+1-x+1}=\frac{16x^2+1+8x+16x^2+9-24x}{16x^2+1+8x-16x^2-9+24x}$$

$$\frac{2x}{2}=\frac{32x^2+10-16x}{32x-8}$$

$$x=\frac{16x^2+5-8x}{16x-4}$$

$$16x^2-4x=16x^2+5-8x$$

$$4x=5$$

$$x=\frac{5}{4}$$

(iii) $$\frac{3x+\sqrt{9x^2-5}}{3x-\sqrt{9x^2-5}}=5$$

Applying componendo and dividendo,

$$\frac{3x+\sqrt{9x^2-5}+3x-\sqrt{9x^2-5}}{3x+\sqrt{9x^2-5}-3x+\sqrt{9x^2-5}}=\frac{5+1}{5-1}$$

$$\frac{6x}{2\sqrt{9x^2-5}}=\frac{6}{4}$$

$$\frac{x}{\sqrt{9x^2-5}}=\frac{1}{2}$$

Squaring both sides,

$$\frac{x^2}{9x^2-5}=\frac{1}{4}$$

$$4x^2=9x^2-5$$

$$5x^2=5$$

$$x^2=1$$

$$x=1$$

Question 12.

If $x = \dfrac{\sqrt{a+3b} + \sqrt{a-3b}}{\sqrt{a+3b} - \sqrt{a-3b}}$, prove that: $3bx^2 - 2ax + 3b = 0$.

Solution:

Since, $\dfrac{x}{1} = \dfrac{\sqrt{a+3b} + \sqrt{a-3b}}{\sqrt{a+3b} - \sqrt{a-3b}}$

Applying componendo and dividendo, we get,

$$\frac{x+1}{x-1} = \frac{\sqrt{a+3b} + \sqrt{a-3b} + \sqrt{a+3b} - \sqrt{a-3b}}{\sqrt{a+3b} + \sqrt{a-3b} - \sqrt{a+3b} + \sqrt{a-3b}}$$

$$\frac{x+1}{x-1} = \frac{2\sqrt{a+3b}}{2\sqrt{a-3b}}$$

Squaring both sides,

$$\frac{x^2+2x+1}{x^2-2x+1} = \frac{a+3b}{a-3b}$$

Again applying componendo and dividendo,

$$\frac{x^2+2x+1+x^2-2x+1}{x^2+2x+1-x^2+2x-1} = \frac{a+3b+a-3b}{a+3b-a+3b}$$

$$\frac{2(x^2+1)}{2(2x)} = \frac{2(a)}{2(3b)}$$

$3b(x^2+1) = 2ax$

$3bx^2 + 3b = 2ax$

$3bx^2 - 2ax + 3b = 0.$

Question 13.

Using the properties of proportion, solve for x,

given $\dfrac{x^4+1}{2x^2} = \dfrac{17}{8}$

Solution:

$$\frac{x^4+1}{2x^2}=\frac{17}{8}$$

Applying componendo and dividendo, we get

$$\frac{x^4+1+2x^2}{x^4+1-2x^2}=\frac{17+8}{17-8}$$

$$\Rightarrow \frac{(x^2)^2+(1)^2+2\times x^2\times 1}{(x^2)^2+(1)^2-2\times x^2\times 1}=\frac{25}{9}$$

$$\Rightarrow \frac{(x^2+1)^2}{(x^2-1)^2}=\frac{5^2}{3^2}$$

$$\Rightarrow \left(\frac{x^2+1}{x^2-1}\right)^2=\left(\frac{5}{3}\right)^2$$

$$\Rightarrow \frac{x^2+1}{x^2-1}=\frac{5}{3}$$

Applying componendo and dividendo, we get

$$\frac{x^2+1+x^2-1}{x^2+1-x^2+1}=\frac{5+3}{5-3}$$

$$\Rightarrow \frac{2x^2}{2}=\frac{8}{2}$$

$$\Rightarrow x^2=4$$

$$\Rightarrow x=\pm 2$$

Question 14.

If $x=\frac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-n}}$, express n in terms of x and m.

Solution:

$$x = \frac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-n}}$$

Applying componendo and dividendo,

$$\frac{x+1}{x-1} = \frac{\sqrt{m+n}+\sqrt{m-n}+\sqrt{m+n}-\sqrt{m-n}}{\sqrt{m+n}+\sqrt{m-n}-\sqrt{m+n}+\sqrt{m-n}}$$

$$\frac{x+1}{x-1} = \frac{2\sqrt{m+n}}{2\sqrt{m-n}}$$

$$\frac{x+1}{x-1} = \frac{\sqrt{m+n}}{\sqrt{m-n}}$$

Squaring both sides,

$$\frac{x^2+2x+1}{x^2-2x+1} = \frac{m+n}{m-n}$$

Applying componendo and dividendo,

$$\frac{x^2+2x+1+x^2-2x+1}{x^2+2x+1-x^2+2x-1} = \frac{m+n+m-n}{m+n-m+n}$$

$$\frac{2x^2+2}{4x} = \frac{2m}{2n}$$

$$\frac{x^2+1}{2x} = \frac{m}{n}$$

$$\frac{x^2+1}{2mx} = \frac{1}{n}$$

$$n = \frac{2mx}{x^2+1}$$

Question 15.

If $\frac{x^3+3xy^2}{3x^2y+y^3} = \frac{m^3+3mn^2}{3m^2n+n^3}$, show that:

$nx = my$.

Solution:

$$\frac{x^3 + 3xy^2}{3x^2y + y^3} = \frac{m^3 + 3mn^2}{3m^2n + n^3}$$

Applying componendo and dividendo,

$$\frac{x^3 + 3xy^2 + 3x^2y + y^3}{x^3 + 3xy^2 - 3x^2y - y^3} = \frac{m^3 + 3mn^2 + 3m^2n + n^3}{m^3 + 3mn^2 - 3m^2n - n^3}$$

$$\frac{(x+y)^3}{(x-y)^3} = \frac{(m+n)^3}{(m-n)^3}$$

$$\frac{x+y}{x-y} = \frac{m+n}{m-n}$$

Applying componendo and dividendo,

$$\frac{x+y+x-y}{x+y-x+y} = \frac{m+n+m-n}{m+n-m+n}$$

$$\frac{2x}{2y} = \frac{2m}{2n}$$

$$\frac{x}{y} = \frac{m}{n}$$

$$nx = my$$

Exercise 7D

Question 1.

If a: b = 3: 5, find:
(10a + 3b): (5a + 2b)
Solution:

Given, $\frac{a}{b} = \frac{3}{5}$

$$\frac{10a + 3b}{5a + 2b}$$

$$= \frac{10\left(\frac{a}{b}\right) + 3}{5\left(\frac{a}{b}\right) + 2}$$

$$= \frac{10\left(\frac{3}{5}\right) + 3}{5\left(\frac{3}{5}\right) + 2}$$

$$= \frac{6 + 3}{3 + 2}$$

$$= \frac{9}{5}$$

Question 2.
If 5x + 6y: 8x + 5y = 8: 9, find x: y.

Solution:

$$\frac{5x + 6y}{8x + 5y} = \frac{8}{9}$$

$$45x + 54y = 64x + 40y$$

$$64x - 45x = 54y - 40y$$

$$19x = 14y$$

$$\frac{x}{y} = \frac{14}{19}$$

Question 3.

If (3x – 4y): (2x – 3y) = (5x – 6y): (4x – 5y), find x: y.
Solution:

$$(3x - 4y) : (2x - 3y) = (5x - 6y) : (4x - 5y)$$

$$\frac{3x - 4y}{2x - 3y} = \frac{5x - 6y}{4x - 5y}$$

Applying componendo and dividendo,

$$\frac{3x - 4y + 2x - 3y}{3x - 4y - 2x + 3y} = \frac{5x - 6y + 4x - 5y}{5x - 6y - 4x + 5y}$$

$$\frac{5x - 7y}{x - y} = \frac{9x - 11y}{x - y}$$

$$5x - 7y = 9x - 11y$$
$$11y - 7y = 9x - 5x$$
$$4y = 4x$$
$$\frac{x}{y} = \frac{1}{1}$$
$$x : y = 1 : 1$$

Question 4.
Find the:
(i) duplicate ratio of $2\sqrt{2} : 3\sqrt{5}$
(ii) triplicate ratio of 2a: 3b
(iii) sub-duplicate ratio of $9x^2a^4 : 25y^6b^2$
(iv) sub-triplicate ratio of 216: 343
(v) reciprocal ratio of 3: 5
(vi) ratio compounded of the duplicate ratio of 5: 6, the reciprocal ratio of 25: 42 and the sub-duplicate ratio of 36: 49.

Solution:

(i) Duplicate ratio of $2\sqrt{2} : 3\sqrt{5} = \left(2\sqrt{2}\right)^2 : \left(3\sqrt{5}\right)^2 = 8 : 45$

(ii) Triplicate ratio of 2a: 3b = $(2a)^3 : (3b)^3 = 8a^3 : 27b^3$
(iii) Sub-duplicate ratio of $9x^2a^4 : 25y^6b^2 = \sqrt{9x^2a^4} : \sqrt{25y^6b^2} = 3xa^2 : 5y^3b$

(iv) Sub-triplicate ratio of 216: 343 = $\sqrt[3]{216} : \sqrt[3]{343} = 6 : 7$
(v) Reciprocal ratio of 3: 5 = 5: 3
(vi) Duplicate ratio of 5: 6 = 25: 36
Reciprocal ratio of 25: 42 = 42: 25
Sub-duplicate ratio of 36: 49 = 6: 7
Required compound ratio = $\frac{25 \times 42 \times 6}{36 \times 25 \times 7} = 1 : 1$

Question 5.
Find the value of x, if:
(i) (2x + 3): (5x – 38) is the duplicate ratio of $\sqrt{5} : \sqrt{6}$
(ii) (2x + 1): (3x + 13) is the sub-duplicate ratio of 9: 25.
(iii) (3x – 7): (4x + 3) is the sub-triplicate ratio of 8: 27.

Solution:

(i) (2x + 3): (5x - 38) is the duplicate ratio of $\sqrt{5} : \sqrt{6}$

Duplicate ratio of $\sqrt{5} : \sqrt{6} = 5 : 6$

$$\frac{2x+3}{5x-38} = \frac{5}{6}$$

$$12x + 18 = 25x - 190$$

$$25x - 12x = 190 + 18$$

$$13x = 208$$

$$x = \frac{208}{13} = 16$$

(ii) (2x + 1): (3x + 13) is the sub-duplicate ratio of 9: 25

Sub-duplicate ratio of 9: 25 = 3: 5

$$\frac{2x+1}{3x+13} = \frac{3}{5}$$

$$10x + 5 = 9x + 39$$

$$10x - 9x = 39 - 5$$

$$x = 34$$

(iii) (3x - 7): (4x + 3) is the sub-triplicate ratio of 8: 27

Sub-triplicate ratio of 8: 27 = 2: 3

$$\frac{3x-7}{4x+3} = \frac{2}{3}$$

$$9x - 21 = 8x + 6$$

$$9x - 8x = 6 + 21$$

$$x = 27$$

Question 6.

What quantity must be added to each term of the ratio x: y so that it may become equal to c: d?

Solution:

Let the required quantity which is to be added be p.

Then, we have:

$$\frac{x+p}{y+p}=\frac{c}{d}$$
$$dx+pd=cy+cp$$
$$pd-cp=cy-dx$$
$$p(d-c)=cy-dx$$
$$p=\frac{cy-dx}{d-c}$$

Question 7.
A woman reduces her weight in the ratio 7 : 5. What does her weight become if originally it was 84 kg?

Solution:

Let the reduced weight be x.
Original weight = 84 kg
Thus, we have
$$84:x=7:5$$
$$\Rightarrow \frac{84}{x}=\frac{7}{5}$$
$$\Rightarrow 84\times 5=7\times x$$
$$\Rightarrow x=\frac{84\times 5}{7}$$
$$\Rightarrow x=60$$
Thus, her reduced weight is 60 kg.

Question 8.
If $15(2x^2 - y^2) = 7xy$, find x: y; if x and y both are positive.

Solution:

$$15(2x^2-y^2)=7xy$$
$$\frac{2x^2-y^2}{xy}=\frac{7}{15}$$
$$\frac{2x}{y}-\frac{y}{x}=\frac{7}{15}$$
Let $\frac{x}{y}=a$

$\therefore 2a - \frac{1}{a} = \frac{7}{15}$

$\frac{2a^2 - 1}{a} = \frac{7}{15}$

$30a^2 - 15 = 7a$

$30a^2 - 7a - 15 = 0$

$30a^2 - 25a + 18a - 15 = 0$

$5a(6a - 5) + 3(6a - 5) = 0$

$(6a - 5)(5a + 3) = 0$

$a = \frac{5}{6}, -\frac{3}{5}$

But, a cannot be negative.

$\therefore a = \frac{5}{6}$

$\Rightarrow \frac{x}{y} = \frac{5}{6}$

$\Rightarrow x : y = 5 : 6$

Question 9.

Find the:

(i) fourth proportional to $2xy$, x^2 and y^2.

(ii) third proportional to $a^2 - b^2$ and $a + b$.

(iii) mean proportional to $(x - y)$ and $(x^3 - x^2y)$.

Solution:

(i) Let the fourth proportional to $2xy$, x^2 and y^2 be n.

$\Rightarrow 2xy : x^2 = y^2 : n$

$\Rightarrow 2xy \times n = x^2 \times y^2$

$\Rightarrow n = \frac{x^2y^2}{2xy} = \frac{xy}{2}$

(ii) Let the third proportional to $a^2 - b^2$ and $a + b$ be n.

$\Rightarrow a^2 - b^2$, $a + b$ and n are in continued proportion.

$\Rightarrow a^2 - b^2 : a + b = a + b : n$

$$\Rightarrow n = \frac{(a+b)^2}{a^2 - b^2} = \frac{(a+b)^2}{(a+b)(a-b)} = \frac{a+b}{a-b}$$

(iii) Let the mean proportional to $(x - y)$ and $(x^3 - x^2y)$ be n.

$\Rightarrow (x - y), n, (x^3 - x^2y)$ are in continued proportion

$\Rightarrow (x - y) : n = n : (x^3 - x^2y)$

$\Rightarrow n^2 = (x - y)(x^3 - x^2y)$

$\Rightarrow n^2 = x^2(x - y)(x - y)$

$\Rightarrow n^2 = x^2(x - y)^2$

$\Rightarrow n = x(x - y)$

Question 10.
Find two numbers such that the mean proportional between them is 14 and third proportional to them is 112.

Solution:

Let the required numbers be a and b.
Given, 14 is the mean proportional between a and b.

$\Rightarrow a : 14 = 14 : b$

$\Rightarrow ab = 196$

$\Rightarrow a = \frac{196}{b}$ ...(1)

Also, given, third proportional to a and b is 112.

$\Rightarrow a : b = b : 112$

$\Rightarrow b^2 = 112a$ ...(2)

Using (1), we have:

$b^2 = 112 \times \frac{196}{b}$

$b^3 = (14)^3(2)^3$

$b = 28$

From (1),

$a = \frac{196}{28} = 7$

Thus, the two numbers are 7 and 28.

Question 11.
If x and y be unequal and x: y is the duplicate ratio of x + z and y + z, prove that z is mean proportional between x and y.

Solution:

Given, $\frac{x}{y} = \frac{(x+z)^2}{(y+z)^2}$

$x(y^2 + z^2 + 2yz) = y(x^2 + z^2 + 2xz)$

$xy^2 + xz^2 + 2xyz = x^2y + yz^2 + 2xyz$

$xy^2 + xz^2 = x^2y + yz^2$

$xy^2 - x^2y = yz^2 - xz^2$

$xy(y - x) = z^2(y - x)$

$xy = z^2$

Hence, z is mean proportional between x and y.

Question 12.
If $x = \frac{2ab}{a+b}$, find the value of $\frac{x+a}{x-a} + \frac{x+b}{x-b}$.

Solution:

$x = \frac{2ab}{a+b}$

$\frac{x}{a} = \frac{2b}{a+b}$

Applying componendo and dividendo,

$\frac{x+a}{x-a} = \frac{2b+a+b}{2b-a-b}$

$\frac{x+a}{x-a} = \frac{3b+a}{b-a}$ ... (1)

Also, $x = \frac{2ab}{a+b}$

$\frac{x}{b} = \frac{2a}{a+b}$

Applying componendo and dividendo,

$$\frac{x+b}{x-b}=\frac{2a+a+b}{2a-a-b}$$

$$\frac{x+b}{x-b}=\frac{3a+b}{a-b} \quad \ldots (2)$$

From (1) and (2),

$$\frac{x+a}{x-a}+\frac{x+b}{x-b}=\frac{3b+a}{b-a}+\frac{3a+b}{a-b}$$

$$\frac{x+a}{x-a}+\frac{x+b}{x-b}=\frac{-3b-a+3a+b}{a-b}$$

$$\frac{x+a}{x-a}+\frac{x+b}{x-b}=\frac{2a-2b}{a-b}=2$$

Question 13.
If (4a + 9b) (4c − 9d) = (4a − 9b) (4c + 9d), prove that:
a: b = c: d.

Solution:

Given, $\frac{4a+9b}{4a-9b}=\frac{4c+9d}{4c-9d}$

Applying componendo and dividendo,

$$\frac{4a+9b+4a-9b}{4a+9b-4a+9b}=\frac{4c+9d+4c-9d}{4c+9d-4c+9d}$$

$$\frac{8a}{18b}=\frac{8c}{18d}$$

$$\frac{a}{b}=\frac{c}{d}$$

Question 14.

If $\frac{a}{b}=\frac{c}{d}$, show that:

$(a+b):(c+d)=\sqrt{a^2+b^2}:\sqrt{c^2+d^2}$

Solution:

$$\text{Let } \frac{a}{b} = \frac{c}{d} = k(\text{say})$$
$$\Rightarrow a = bk,\ c = dk$$
$$\text{L.H.S.} = \frac{a+b}{c+d}$$
$$= \frac{bk+b}{dk+d}$$
$$= \frac{b(k+1)}{d(k+1)}$$
$$= \frac{b}{d}$$
$$\text{R.H.S.} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$$
$$= \frac{\sqrt{(bk)^2+b^2}}{\sqrt{(dk)^2+d^2}}$$
$$= \frac{\sqrt{b^2(k^2+1)}}{\sqrt{d^2(k^2+1)}}$$
$$= \frac{\sqrt{b^2}}{\sqrt{d^2}}$$
$$= \frac{b}{d}$$
$$\therefore \text{L.H.S.} = \text{R.H.S}$$

Question 15.
There are 36 members in a student council in a school and the ratio of the number of boys to the number of girls is 3: 1. How any more girls should be added to the council so that the ratio of the number of boys to the number of girls may be 9: 5?

Solution:

Ratio of number of boys to the number of girls = 3: 1
Let the number of boys be 3x and number of girls be x.
3x + x = 36
4x = 36
x = 9

∴ Number of boys = 27
Number of girls = 9

Le n number of girls be added to the council.
From given information, we have:

$$\frac{27}{9+n} = \frac{9}{5}$$
$$135 = 81 + 9n$$
$$9n = 54$$
$$n = 6$$

Thus, 6 girls are added to the council.

Question 16.
If 7x – 15y = 4x + y, find the value of x: y. Hence, use componendo and dividend to find the values of:

(i) $\frac{9x+5y}{9x-5y}$

(ii) $\frac{3x^2+2y^2}{3x^2-2y^2}$

Solution:

$$7x - 15y = 4x + y$$
$$7x - 4x = y + 15y$$
$$3x = 16y$$
$$\frac{x}{y} = \frac{16}{3}$$

(i) $\frac{x}{y} = \frac{16}{3}$

$\Rightarrow \frac{9x}{5y} = \frac{144}{15}$ $\left(\text{Multiplying both sides by } \frac{9}{5}\right)$

$\Rightarrow \frac{9x+5y}{9x-5y} = \frac{144+15}{144-15}$ (Applying componendo and dividendo)

$\Rightarrow \frac{9x+5y}{9x-5y} = \frac{159}{129} = \frac{53}{43}$

(ii) $\frac{x}{y} = \frac{16}{3}$

$\Rightarrow \frac{x^2}{y^2} = \frac{256}{9}$

$\Rightarrow \frac{3x^2}{2y^2} = \frac{768}{18} = \frac{128}{3}$ $\left(\text{Multiplying both sides by } \frac{3}{2}\right)$

$\Rightarrow \frac{3x^2 + 2y^2}{3x^2 - 2y^2} = \frac{128 + 3}{128 - 3}$ (Applying componendo and dividendo)

$\Rightarrow \frac{3x^2 + 2y^2}{3x^2 - 2y^2} = \frac{131}{125}$

Question 17.

If $\frac{4m + 3n}{4m - 3n} = \frac{7}{4}$, use properties of proportion to find:

(i) m: n

(ii) $\frac{2m^2 - 11n^2}{2m^2 + 11n^2}$

Solution:

(i) Given, $\frac{4m + 3n}{4m - 3n} = \frac{7}{4}$

Applying componendo and dividendo,

$\frac{4m + 3n + 4m - 3n}{4m + 3n - 4m + 3n} = \frac{7 + 4}{7 - 4}$

$\frac{8m}{6n} = \frac{11}{3}$

$\frac{m}{n} = \frac{11}{4}$

(ii) $\frac{m}{n} = \frac{11}{4}$

$\frac{m^2}{n^2} = \frac{121}{16}$

$$\frac{2m^2}{11n^2} = \frac{2 \times 121}{11 \times 16} \quad \left(\text{Multiplying both sides by } \frac{2}{11}\right)$$

$$\frac{2m^2}{11n^2} = \frac{11}{8}$$

$$\frac{2m^2 + 11n^2}{2m^2 - 11n^2} = \frac{11 + 8}{11 - 8} \quad (\text{Applying componendo and dividendo})$$

$$\frac{2m^2 + 11n^2}{2m^2 - 11n^2} = \frac{19}{3}$$

$$\frac{2m^2 - 11n^2}{2m^2 + 11n^2} = \frac{3}{19} \quad (\text{Applying invertendo})$$

Question 18.

If x, y, z are in continued proportion, prove that $\frac{(x+y)^2}{(y+z)^2} = \frac{x}{z}$.

Solution:

$\because$ x, y, z are in continued proportion.

$\therefore \frac{x}{y} = \frac{y}{z} \Rightarrow y^2 = zx \ldots.(1)$

Therefore,

$$\frac{x+y}{y} = \frac{y+z}{z} \quad (\textit{By componendo})$$

$$\Rightarrow \frac{x+y}{y+z} = \frac{y}{z} \quad (\text{By alternendo})$$

$$\Rightarrow \frac{(x+y)^2}{(y+z)^2} = \frac{y^2}{z^2} \quad (\text{squaring both sides})$$

$$\Rightarrow \frac{(x+y)^2}{(y+z)^2} = \frac{zx}{z^2} \quad [\text{from}(1)]$$

$$\Rightarrow \frac{(x+y)^2}{(y+z)^2} = \frac{x}{z}$$

Hence Proved.

Question 19.

Given $x = \dfrac{\sqrt{a^2 + b^2} + \sqrt{a^2 - b^2}}{\sqrt{a^2 + b^2} - \sqrt{a^2 - b^2}}$.

Use componendo and dividendo to prove that $b^2 = \dfrac{2a^2x}{x^2 + 1}$.

Solution:

$$x = \frac{\sqrt{a^2 + b^2} + \sqrt{a^2 - b^2}}{\sqrt{a^2 + b^2} - \sqrt{a^2 - b^2}}$$

By componendo and dividendo,

$$\frac{x+1}{x-1} = \frac{\sqrt{a^2 + b^2} + \sqrt{a^2 - b^2} + \sqrt{a^2 + b^2} - \sqrt{a^2 - b^2}}{\sqrt{a^2 + b^2} + \sqrt{a^2 - b^2} - \sqrt{a^2 + b^2} + \sqrt{a^2 - b^2}}$$

$$\frac{x+1}{x-1} = \frac{2\sqrt{a^2 + b^2}}{2\sqrt{a^2 - b^2}}$$

Squaring both sides,

$$\frac{x^2 + 2x + 1}{x^2 - 2x + 1} = \frac{a^2 + b^2}{a^2 - b^2}$$

By componendo and dividendo,

$$\frac{(x^2 + 2x + 1) + (x^2 - 2x + 1)}{(x^2 + 2x + 1) - (x^2 - 2x + 1)} = \frac{(a^2 + b^2) + (a^2 - b^2)}{(a^2 + b^2) - (a^2 - b^2)}$$

$$\Rightarrow \frac{2(x^2 + 1)}{4x} = \frac{2a^2}{2b^2}$$

$$\Rightarrow \frac{x^2 + 1}{2x} = \frac{a^2}{b^2}$$

$$\Rightarrow b^2 = \frac{2a^2x}{x^2 + 1}$$

Hence Proved.

Question 20.

If $\frac{x^2+y^2}{x^2-y^2} = 2\frac{1}{8}$, find:

(i) $\frac{x}{y}$

(ii) $\frac{x^3+y^3}{x^3-y^3}$

Solution:

(i) Given, $\frac{x^2+y^2}{x^2-y^2} = 2\frac{1}{8}$

$\frac{x^2+y^2}{x^2-y^2} = \frac{17}{8}$

Applying componendo and dividendo,

$\frac{x^2+y^2+x^2-y^2}{x^2+y^2-x^2+y^2} = \frac{17+8}{17-8}$

$\frac{2x^2}{2y^2} = \frac{25}{9}$

$\frac{x^2}{y^2} = \frac{25}{9}$

$\frac{x}{y} = \frac{5}{3} = 1\frac{2}{3}$

(ii) $\frac{x^3+y^3}{x^3-y^3}$

$= \frac{\left(\frac{x}{y}\right)^3 + 1}{\left(\frac{x}{y}\right)^3 - 1}$

$= \frac{\left(\frac{5}{3}\right)^3 + 1}{\left(\frac{5}{3}\right)^3 - 1}$

$= \frac{\frac{125}{27} + 1}{\frac{125}{27} - 1}$

$$= \frac{\frac{125+27}{27}}{\frac{125-27}{27}}$$

$$= \frac{125+27}{125-27}$$

$$= \frac{76}{49} = 1\frac{27}{49}$$

Question 21.

Using componendo and dividendo find the value of x:

$$\frac{\sqrt{3x+4}+\sqrt{3x-5}}{\sqrt{3x+4}-\sqrt{3x-5}} = 9$$

Solution:

$$\frac{\sqrt{3x+4}+\sqrt{3x-5}}{\sqrt{3x+4}-\sqrt{3x-5}} = \frac{9}{1}$$

Applying componendo and dividendo, we have

$$\frac{\sqrt{3x+4}+\sqrt{3x-5}+\sqrt{3x+4}-\sqrt{3x-5}}{\sqrt{3x+4}+\sqrt{3x-5}-\sqrt{3x+4}+\sqrt{3x-5}} = \frac{9+1}{9-1}$$

$$\Rightarrow \frac{2\sqrt{3x+4}}{2\sqrt{3x-5}} = \frac{10}{8}$$

$$\Rightarrow \frac{\sqrt{3x+4}}{\sqrt{3x-5}} = \frac{5}{4}$$

Squaring both sides, we have

$$\frac{3x+4}{3x-5} = \frac{25}{16}$$

$$\Rightarrow 16(3x+4) = 25(3x-5)$$

$$\Rightarrow 48x + 64 = 75x - 125$$

$$\Rightarrow 75x - 48x = 64 + 125$$

$$\Rightarrow 27x = 189$$

$$\Rightarrow x = \frac{189}{27}$$

$$\Rightarrow x = 7$$

Question 22.

If $x = \dfrac{\sqrt{a+1} + \sqrt{a-1}}{\sqrt{a+1} + \sqrt{a-1}}$, using properties of proportion,
show that:
$x^2 - 2ax + 1 = 0$

Solution:

Given that, $x = \dfrac{\sqrt{a+1} + \sqrt{a-1}}{\sqrt{a+1} - \sqrt{a-1}}$

By applying Componendo-Dividendo,

$$\frac{x+1}{x-1} = \frac{\left(\sqrt{a+1} + \sqrt{a-1}\right) + \left(\sqrt{a+1} - \sqrt{a-1}\right)}{\left(\sqrt{a+1} + \sqrt{a-1}\right) - \left(\sqrt{a+1} - \sqrt{a-1}\right)}$$

$$\Rightarrow \frac{x+1}{x-1} = \frac{2\sqrt{a+1}}{2\sqrt{a-1}}$$

$$\Rightarrow \frac{x+1}{x-1} = \frac{\sqrt{a+1}}{\sqrt{a-1}}$$

Squaring both the sides of the equation, we have,

$$\Rightarrow \left(\frac{x+1}{x-1}\right)^2 = \frac{a+1}{a-1}$$

$$\Rightarrow (x+1)^2 (a-1) = (x-1)^2 (a+1)$$

$$\Rightarrow \left(x^2+2x+1\right)(a-1) = \left(x^2-2x+1\right)(a+1)$$

$$\Rightarrow a\left(x^2+2x+1\right) - \left(x^2+2x+1\right) = a\left(x^2-2x+1\right) + \left(x^2-2x+1\right)$$

$$\Rightarrow 4ax = 2x^2 + 2$$

$$\Rightarrow 2ax = x^2 + 1$$

$$\Rightarrow x^2 - 2ax + 1 = 0$$

Question 23.

Given $\dfrac{x^3 + 12x}{6x^2 + 8} = \dfrac{y^3 + 27y}{9y^2 + 27}$.

Using componendo and dividendo, find $x : y$.

Solution:

$$\frac{x^3+12x}{6x^2+8}=\frac{y^3+27y}{9y^2+27}$$

Applying componendo and dividendo, we get

$$\frac{x^3+12x+6x^2+8}{x^3+12x-6x^2-8}=\frac{y^3+27y+9y^2+27}{y^3+27y-9y^2-27}$$

$$\Rightarrow \frac{x^3+3(1)(4)x+3(1)(2)x^2+2^3}{x^3+3(1)(4)x-3(1)(2)x^2-2^3}=\frac{y^3+3(1)(9)y+3(1)(3)y^2+3^3}{y^3+3(1)(9)y-3(1)(3)y^2-3^3}$$

$$\Rightarrow \frac{x^3+3(1)(4)x+3(1)(2)x^2+2^3}{x^3-3(1)(2)x^2+3(1)(4)x-2^3}=\frac{y^3+3(1)(9)y+3(1)(3)y^2+3^3}{y^3-3(1)(3)y^2+3(1)(9)y-3^3}$$

$$\Rightarrow \frac{(x+2)^3}{(x-2)^3}=\frac{(y+3)^3}{(y-3)^3}$$

$$\Rightarrow \frac{x+2}{x-2}=\frac{y+3}{y-3}$$

Again applying componendo and dividendo, we get

$$\frac{x+2+x-2}{x+2-x+2}=\frac{y+3+y-3}{y+3-y+3}$$

$$\Rightarrow \frac{2x}{4}=\frac{2y}{6}$$

$$\Rightarrow \frac{x}{2}=\frac{y}{3}$$

Applying alternendo, we get

$$\frac{x}{y}=\frac{2}{3}$$

$$\Rightarrow x:y=2:3$$

Question 24.

$$\text{Let } \frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$$

$$\Rightarrow x=ak, y=bk, z=ck$$

$$\text{L.H.S.}=\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}$$

$$= \frac{(ak)^3}{a^3} + \frac{(bk)^3}{b^3} + \frac{(ck)^3}{c^3}$$
$$= \frac{a^3k^3}{a^3} + \frac{b^3k^3}{b^3} + \frac{c^3k^3}{c^3}$$
$$= k^3 + k^3 + k^3$$
$$= 3k^3$$
$$\text{R.H.S.} = \frac{3xyz}{abc}$$
$$= \frac{3(ak)(bk)(ck)}{abc}$$
$$= 3k^3$$
$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$
$$\text{i.e. } \frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{z^3}{c^3} = \frac{3xyz}{abc}$$

Question 25.

Given that b is the mean proportion between a and c.

$$\therefore \frac{a}{b} = \frac{b}{c} = k$$
$$\Rightarrow b = ck \;;\quad a = bk = (ck)k = ck^2$$
$$\text{L.H.S} = \frac{a^4 + a^2b^2 + b^4}{b^4 + b^2c^2 + c^4}$$
$$= \frac{(ck^2)^4 + (ck^2)^2(ck)^2 + (ck)^4}{(ck)^4 + (ck)^2c^2 + c^4}$$
$$= \frac{c^4k^8 + c^4k^6 + c^4k^4}{c^4k^4 + c^4k^2 + c^4}$$
$$= \frac{c^4k^4(k^4 + k^2 + 1)}{c^4(k^4 + k^2 + 1)}$$
$$= k^4 \quad \ldots. \text{ (i)}$$
$$\text{R. H. S} = \frac{a^2}{c^2} = \frac{(ck^2)^2}{c^2} = \frac{c^2k^4}{c^2} = k^4 \quad \ldots \text{ (ii)}$$

From (i) and (ii), we get

L.H.S = R.H.S

$\Rightarrow \frac{a^4 + a^2b^2 + b^4}{b^4 + b^2c^2 + c^4} = \frac{a^2}{c^2}$

Hence proved.

Question 26.

$\frac{7m + 2n}{7m - 2n} = \frac{5}{3}$

Applying Componendo and Dividendo, we get

$\frac{7m + 2n + 7m - 2n}{7m + 2n - 7m + 2n} = \frac{5 + 3}{5 - 3}$

$\Rightarrow \frac{14m}{4n} = \frac{8}{2}$

$\Rightarrow \frac{m}{n} = \frac{8 \times 4}{2 \times 14}$

$\Rightarrow \frac{m}{n} = \frac{8}{7}$

$\Rightarrow m : n = 8 : 7$

ii.

From (i),

$\frac{m}{n} = \frac{8}{7}$

$\Rightarrow \frac{m^2}{n^2} = \frac{64}{49}$

Applying Componendo and Dividendo, we get

$\frac{m^2 + n^2}{m^2 - n^2} = \frac{64 + 49}{64 - 49}$

$\Rightarrow \frac{m^2 + n^2}{m^2 - n^2} = \frac{64 + 49}{64 - 49}$

$\Rightarrow \frac{m^2 + n^2}{m^2 - n^2} = \frac{113}{15} = 7\frac{8}{15}$

Question 27.

i. (2x2 - 5y2): xy = 1: 3

$$\Rightarrow \frac{2x^2 - 5y^2}{xy} = \frac{1}{3}$$

$$\Rightarrow \frac{2x}{y} - \frac{5y}{x} = \frac{1}{3}$$

Put $\frac{x}{y} = a$, we get

$$\Rightarrow 2a - 5\frac{1}{a} = \frac{1}{3}$$

$$\Rightarrow 3(2a^2 - 5) = a$$

$$\Rightarrow 6a^2 - a - 15 = 0$$

$$\Rightarrow 6a^2 + 9a - 10a - 15 = 0$$

$$\Rightarrow 3a(2a + 3) - 5(2a + 3) = 0$$

$$\Rightarrow (2a + 3)(3a - 5) = 0$$

$$\Rightarrow (2a + 3) = 0 \text{ or } (3a - 5) = 0$$

$$\Rightarrow a = -\frac{3}{2} \text{ or } a = \frac{5}{3}$$

$a = -\frac{3}{2}$ is not acceptable, as x and y both are positive.

$$\therefore a = \frac{5}{3} \Rightarrow \frac{x}{y} = \frac{5}{3}$$

$$\Rightarrow x : y = 5 : 3$$

ii.

$$16\left(\frac{a - x}{a + x}\right)^3 = \frac{a + x}{a - x}$$

$$\Rightarrow 16 = \left(\frac{a + x}{a - x}\right)^4$$

$$\Rightarrow (2)^4 = \left(\frac{a + x}{a - x}\right)^4$$

$$\Rightarrow \frac{a + x}{a - x} = \pm 2$$

$$\Rightarrow \frac{a+x}{a-x} = \frac{2}{1} \quad \text{or} \quad \frac{a+x}{a-x} = \frac{-2}{1}$$

Applying Componendo and Dividendo, we get

$$\Rightarrow \frac{a+x+a-x}{a+x-a+x} = \frac{3}{1} \quad \text{or} \quad \frac{a+x+a-x}{a+x-a+x} = \frac{-1}{-3}$$

$$\Rightarrow \frac{2a}{2x} = 3 \quad \text{or} \quad \frac{2a}{2x} = \frac{1}{3}$$

$$\Rightarrow x = \frac{a}{3} \text{ or } x = 3a$$

Question 28.

If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove that:

$$\frac{ax-by}{(a+b)(x-y)} + \frac{by-cz}{(b+c)(y-z)} + \frac{cz-ax}{(c+a)(z-x)} = 3$$

Solution:

Let $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k(\text{say})$

$\Rightarrow x = ak, y = bk, z = ck$

L.H.S.

$$= \frac{ax-by}{(a+b)(x-y)} + \frac{by-cz}{(b+c)(y-z)} + \frac{cz-ax}{(c+a)(z-x)}$$

$$= \frac{a(ak)-b(bk)}{(a+b)(ak-bk)} + \frac{b(bk)-c(ck)}{(b+c)(bk-ck)} + \frac{c(ck)-a(ak)}{(c+a)(ck-ak)}$$

$$= \frac{k(a^2-b^2)}{k(a+b)(a-b)} + \frac{k(b^2-c^2)}{k(b+c)(b-c)} + \frac{k(c^2-a^2)}{k(c+a)(c-a)}$$

$$= \frac{k(a^2-b^2)}{k(a^2-b^2)} + \frac{k(b^2-c^2)}{k(b^2-c^2)} + \frac{k(c^2-a^2)}{k(c^2-a^2)}$$

$= 1 + 1 + 1 = 3 = \text{R.H.S.}$

Question 29.

If q is the mean proportional between p and r, prove that:

$$\frac{p^3+q^3+r^3}{p^2q^2r^2} = \frac{1}{p^3}+\frac{1}{q^3}+\frac{1}{r^3}.$$

Solution:

Since, q is the mean proportional between p and r,
$q^2 = pr$

$$\text{L.H.S.} = \frac{p^3+q^3+r^3}{p^2q^2r^2}$$
$$= \frac{p^3+(pr)q+r^3}{p^2(pr)r^2}$$
$$= \frac{p^3+pqr+r^3}{p^3r^3}$$
$$= \frac{1}{r^3}+\frac{q}{p^2r^2}+\frac{1}{p^3}$$
$$= \frac{1}{r^3}+\frac{q}{(q^2)^2}+\frac{1}{p^3}$$
$$= \frac{1}{r^3}+\frac{1}{q^3}+\frac{1}{p^3}$$
$$= \text{R.H.S.}$$

Question 30.
If a, b and c are in continued proportion, prove that:
$a : c = (a^2 + b^2) : (b^2 + c^2)$

Solution:

Given, a, b and c are in continued proportion.
$\Rightarrow a : b = b : c$

Let $\frac{a}{b} = \frac{b}{c} = k$ (say)

$\Rightarrow a = bk, b = ck$

$\Rightarrow a = ck^2, b = ck$

Now, L.H.S. $= \frac{a}{c} = \frac{ck^2}{c} = k^2$

$$\text{R.H.S.} = \frac{a^2 + b^2}{b^2 + c^2}$$

$$= \frac{(ck^2)^2 + (ck)^2}{(ck)^2 + c^2}$$

$$= \frac{c^2k^4 + c^2k^2}{c^2k^2 + c^2}$$

$$= \frac{c^2k^2(k^2 + 1)}{c^2(k^2 + 1)}$$

$$= k^2$$

$\therefore$ L.H.S. = R.H.S.